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Strong and weak (1, 3) homotopies on spherical curves and related topics

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1 Introduction

A spherical curve is the image of a generic immersion of a circle into a two dimensional sphere. It is known that any two spherical curves are related by a finite sequence of first, second, and third Reidemeister moves. The first, second, and third Reidemeister moves are denoted by RI, RII, and RIII, respectively.

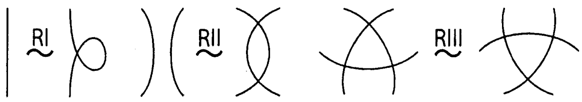

Figure 1: Reidemeister moves

By using the rotation number, we can detect spherical curves, corresponding to a pair, that are related by a finite sequence consisting of RII and RIII. Hence, about three years ago, one of the authors of this paper Takimura, asked the following questions:

Question (1) How to obtain the necessary and sufficient condition that any two spherical curves are related by a finite sequence consisting of RI and RII?

Question (2) How to obtain the necessary and sufficient condition that any two spherical curves are related by a finite sequence consisting of RI and RIII? The solution to Question (1) can be found in [7]. However, Question (2) is still open. However, we would like to mention that there exists a spherical curve that is non-trivial under an equivalence relation by RI and RIII [2]. Here, note that RI can change an isotopy type of a Legendrian knot.

Many studies concerning with Arnold invariants are often not useful for classification problems related to RI.

For Question (2), Ito, one of the authors, considered smaller problems concerning with pairs, (RI, strong RIII) and (RI, weak RIII) (see Definition 1), as a first step.

2 Preliminaries

Definition 1. Reidemeister moves consist of the five types of local replacements in a sufficiently small disk on a 2-sphere. The five types of local replacements are RI, strong RII, weak RII, strong RIII, and weak RIII. Here, strong RII, weak RII, strong RIII, and weak RIII are defined in Fig. 2. RI (resp. RII) increasing double point is denoted by $1a$ (resp. $2b$).

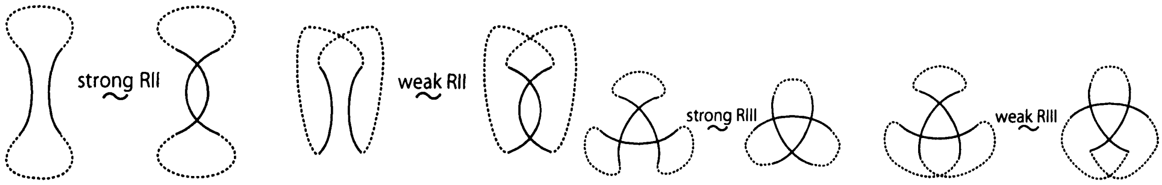

Figure 2: Strong RII and weak RII (left); strong RIII and weak RIII (right)

Similarly, strong RII (resp. weak RII) increasing double point is denoted by $s2b$ (resp. $w2b$). As shown in Fig. 2, a single strong RIII, from the right to the left, is denoted by $s3b$.

3 Summary—several characterizations of trivialities.

In this section, we select known results related to the spherical curve is trivialized by a finite sequence consisting of certain types of Reidemeister moves.

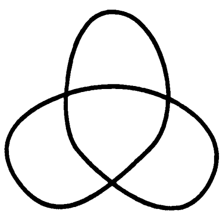

Figure 3: Trefoil projection

Let P be a spherical curve and O the spherical curve with no double points. A spherical curve is the projection image of the standard knot diagram of the trefoil, as shown in Fig. 3.

(1) P and O can be related by a finite sequence consisting of RI if and only if there exists a finite sequence consisting of $1b$ from P to O [3].

(2) P and O can be related by a finite sequence consisting of RII if and only if there exists a finite sequence consisting of $2b$ from P to O [3].

(3) P and O can be related by a finite sequence consisting of RI and RII if and only if there exists a finite sequence consisting of $1b$ and $2b$ from P to O [7] (cf. [3]).

(4) P and O can be related by a finite sequence consisting of RI and weak RII if and only if there exists a finite sequence consisting of $1b$ and $w2b$ from P to O [4].

(5) P and O can be related by a finite sequence consisting of RI and strong RII if and only if there exists a finite sequence consisting of $1b$ and $s2b$ from P to O [4].

(6) P and O can be related by a finite sequence consisting of RI and weak RIII if and only if there exists a finite sequence consisting of $1b$ from P to O [3].

(7) P and O can be related by a finite sequence consisting of RI and strong RIII if and only if there exists a finite sequence consisting of $1b$ and $3b$ from P to O [6].

(8) P and O can be related by a finite sequence consisting of RI and strong RIII if and only if P is a connected sum consisting of a finite number of spherical curves, each of which is O, the curve appearing as ∞, or the trefoil projection [6].

Another characterization of the triviality corresponding to (6) or (8) is obtained by [5] using so-called chord diagrams up to three chords:  , , and .

4 Prime reduced spherical curves up to seven double points

Fig. 4 consists of prime reduced spherical curves up to seven double points. The list of spherical curves is obtained by the Rolfsen table of knot diagrams and flypes. There exist three pairs $(7_6, 7_A)$, $(7_7, 7_B)$, and $(7_5, 7_C)$, where the two spherical curves corresponding to each pair are related by a flype. Two spherical curves are connected by a solid segment labeled as "s" (resp. "w") if they are related by a finite sequence consisting of a single strong RIII (resp. weak RIII) and a finite number of RIs. We would like to mention that 7_4 and 7_B (resp. 7_5 and 7_C) can be related by a finite sequence consisting of RI and strong RIII (resp. weak RIII) via a prime spherical curve with eight double points, which can be seen in Fig. 5 (resp. Fig. 6). For the pairs (7_4, 7_B) and (7_5, 7_C), a pair of knot projections is connected by a dotted arc, as shown in Fig. 4.

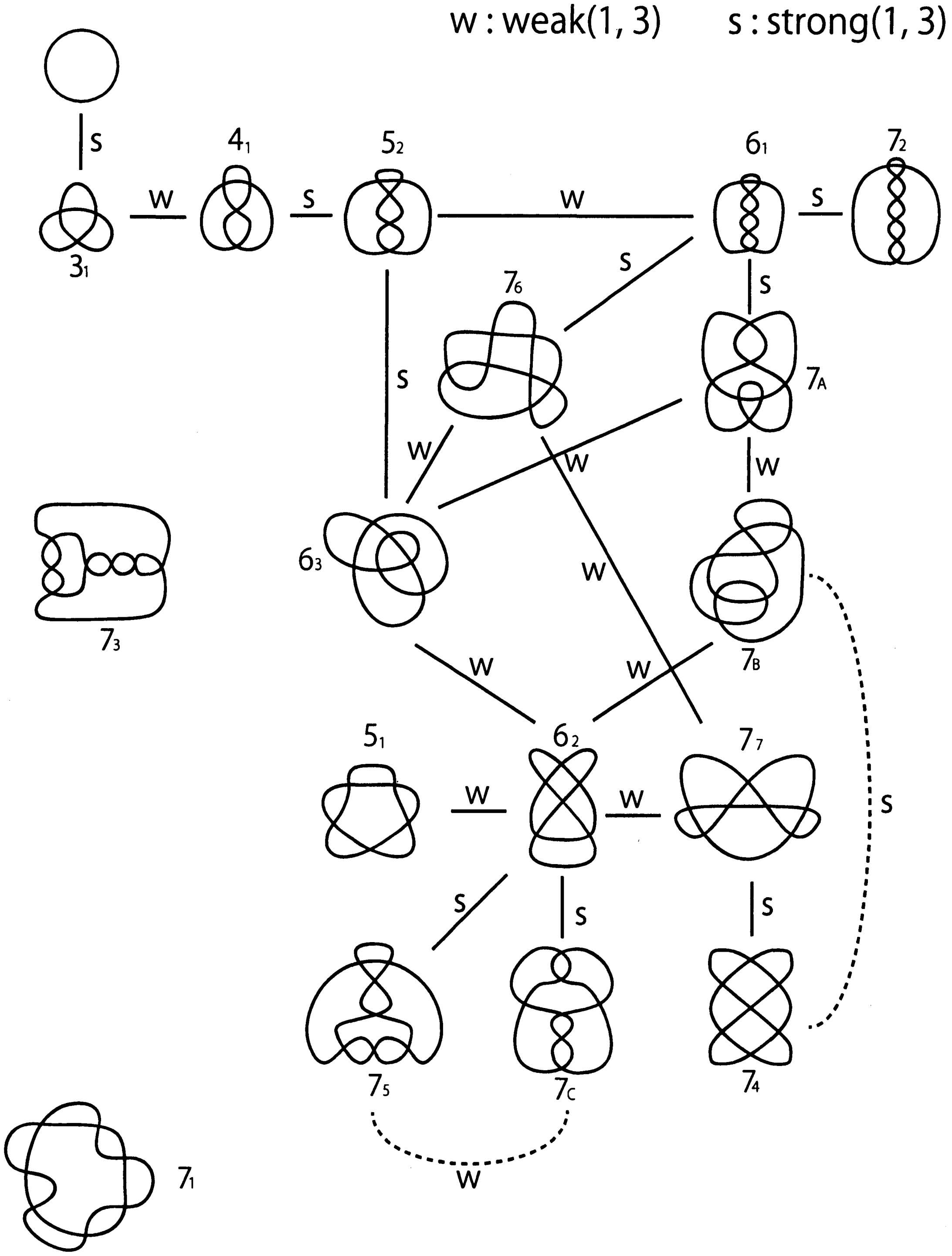

Figure 4: Classification of prime reduced spherical curves up to seven double points

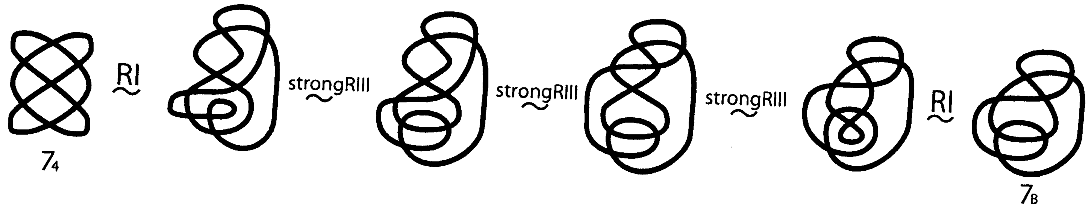

Figure 5: Path between 7_4 and 7_B

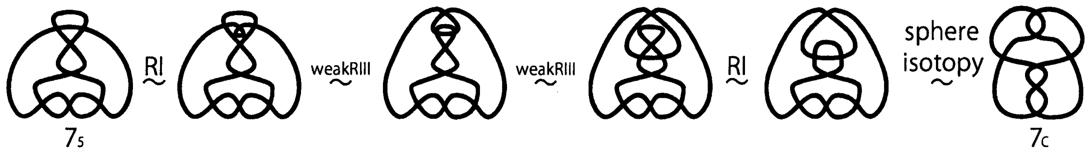

Figure 6: Path between 7_5 and 7_C

References

[1] V. I. Arnol'd, Topological invariants of plane curves and caustics. University Lecture Series, 5. *American Mathematical Society, Providence, RI,* 1994.

[2] T. Hagge and J. Yazinski, On the necessity of Reidemeister move 2 for simplifying immersed planar curves, *Banach Center Publ.* **103** (2014), 101–110.

[3] N. Ito and Y. Takimura, (1, 2) and weak (1, 3) homotopies on knot projections, *J. Knot Theory Ramifications* **22** (2013), 1350085, 14pp.

[4] N. Ito and Y. Takimura, Strong and weak (1, 2) homotopies on knot projections and new invariants, to appear in *Kobe J. Math.*

[5] N. Ito and Y. Takimura, Sub-chord diagrams of knot projections, to appear in *Houston J. Math.*

[6] N. Ito, Y. Takimura, and K. Taniyama, Strong and weak (1, 3) homotopies on knot projections, to appear in *Osaka J. Math.*

[7] M. Khovanov, Doodle groups, *Trans. Amer. Math. Soc.* **349** (1997), 2297–2315.

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